Oscar/Nemo.jl: A Julia package for computer algebra

Claus Fieker, William Hart,
Tommy Hofmann, Fredrik Johansson,
Marek Kaluba

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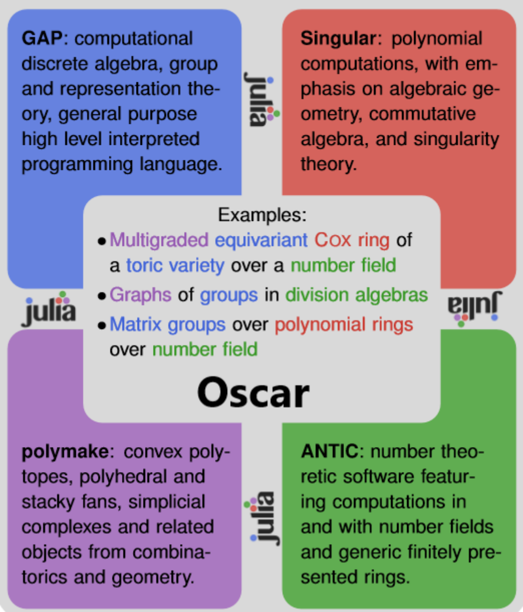
GAP: computational discrete algebra, group and representation theory, general purpose high level interpreted programming language.
Singular: polynomial computations, with emphasis on algebraic geometry, commutative algebra, and singularity theory.
Examples:
• Multigraded equivariant Cox ring of a toric variety over a number field
• Graphs of groups in division algebras
• Matrix groups over polynomial rings over number field
Oscar
polymake: convex polytopes, polyhedral and stacky fans, simplicial complexes and related objects from combinatorics and geometry.
ANTIC: number theoretic software featuring computations in and with number fields and generic finitely presented rings.
julia
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Oscar components : cornerstone systems

- Gap : Group theory (discrete algebra)
- Singular : Polynomials, algebra, geometry
- Polymake : Polyhedreal geometry
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- JuliaInterface : Access to Julia from Gap

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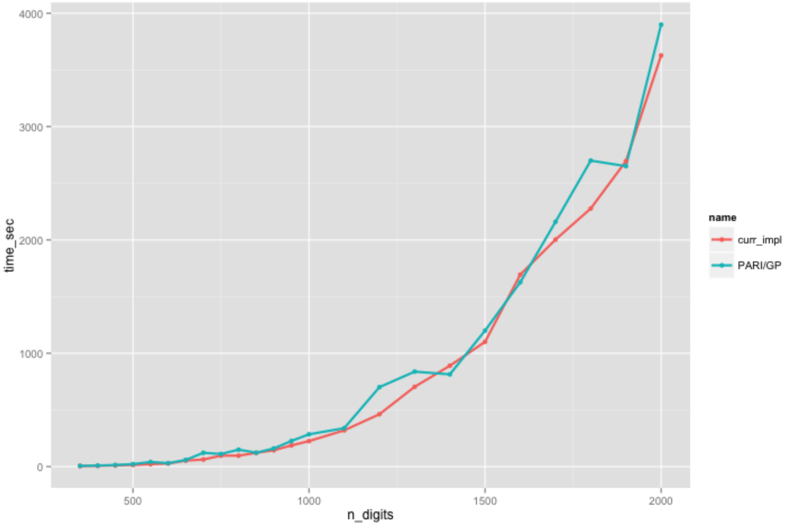
- Multivariate polynomial arithmetic over $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Q}$

Integer factorisation : Quadratic sieve

Table: Quadratic sieve timings

Digits	Pari/GP	Flint (1 core)	Flint (4 cores)
50	0.43	0.55	0.39
59	3.8	3.0	1.7
68	38	21	14
77	257	140	52
83	2200	1500	540

APRCL primality test timings



FFT: Integer and polynomial multiplication

Table: FFT timings

Words	1 core	4 cores	8 cores
110k	0.07s	0.05s	0.05s
360k	0.3s	0.1	0.1s
1.3m	1.1s	0.4s	0.3s
4.6m	4.5s	1.5s	1.0s
26m	28s	9s	6s
120m	140s	48s	33s
500m	800s	240s	150s

Characteristic and minimal polynomial

Table: Charpoly and minpoly timings

Op	Sage 6.9	Pari 2.7.4	Magma 2.21-4	Giac 1.2.2	Flint
Charpoly	0.2s	0.6s	0.06s	0.06s	0.04s
Minpoly	0.07s	>160 hrs	0.05s	0.06s	0.04s

for 80×80 matrix over $\mathbb{Z}$ with entries in $[-20, 20]$ and minpoly of degree 40.

Multivariate multiplication

Table: "Dense" Fateman multiply bench

n	Sage	Singular	Magma	Giac	Piranha	Trip	Flint
5	0.0063s	0.0048s	0.0018s	0.00023s	0.0011s	0.00057s	0.00023s
10	0.51s	0.11s	0.12s	0.0056s	0.029s	0.023s	0.0043s
15	9.1s	1.4s	1.9s	0.11s	0.39s	0.21s	0.045s
20	75s	21s	16s	0.62s	2.9s	2.3s	0.48s
25	474s	156s	98s	2.8s	14s	12s	2.3s
30	1667s	561s	440s	14s	56s	41s	10s

4 variables

Multivariate multiplication

Table: Sparse multiply benchmark

n	Sage	Singular	Magma	Giac	Piranha	Trip	Flint
4	0.0066s	0.0050s	0.0062s	0.0046s	0.0033s	0.0015s	0.0014s
6	0.15s	0.11s	0.080s	0.030s	0.025s	0.016s	0.016s
8	1.6s	0.79s	0.68s	0.28s	0.15s	0.10s	0.10s
10	8s	3.6s	3.0s	1.5s	0.62s	0.40s	0.48s
12	43s	14s	11s	4.8s	2.2s	2.2s	2.0s
14	173s	63s	37s	14s	6.7s	12s	7.2s
16	605s	201s	94s	39s	20s	39s	19s

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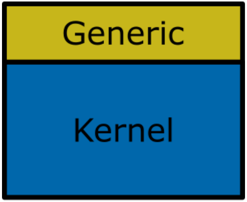
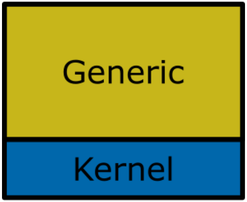
- Easy/efficient C interop

Efficient generics

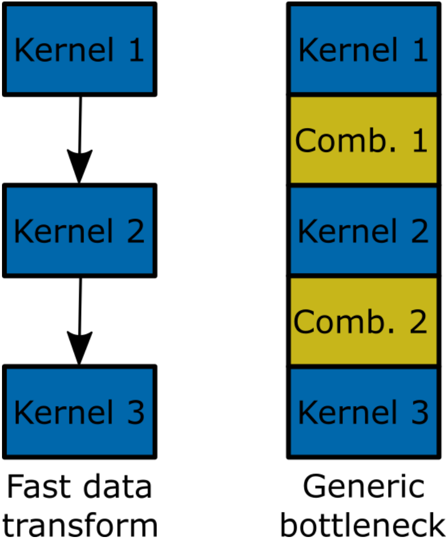


Fast generics



Slow generics

Efficient generics



julia

- JIT compilation : near C performance.
- Designed by mathematically minded people.
- Open Source (MIT License).
- Actively developed since 2009.
- Supports Windows, OSX, Linux, BSD.
- Friendly C/Python-like (imperative) syntax.

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- Integration with Singular.jl

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- Example: Klein j-function is a modular function
- Periodic with period 1, so has Fourier expansion, called a q-expansion:
- $j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$, where $q = exp(2\pi i\tau)$

Modular equation example

- For $n = 2$: $P_2(j(\tau), j(2\tau)) = 0$ for

$$P_2(X,Y) = X^3 - X^2Y2 + 1488X^2Y^2 - 162000X^2 + 1488XY^2$$
$$40773374XY + 8748000000X + Y^2 - 162000Y^2 + 8748000000Y$$
$$- 157464000000000.$$

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- Enter the Weber functions:

$$\mathfrak{f}(\tau) = \frac{\eta^2(\tau)}{\eta(\tau/2)\eta(2\tau)},$$

$$\mathfrak{f}_1(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)},$$

$$\mathfrak{f}_2(\tau) = \frac{\sqrt{2}\eta(2\tau)}{\eta(\tau)}.$$

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- For example, for $n = 13$ we have $\Phi_{13}(\mathfrak{f}(\tau), \mathfrak{f}(13\tau)) = 0$ where

$$\Phi_{13}(X, Y) = X^{14} - X^{13}Y^{13} + 13X^{12}Y^2 + 52X^{10}Y^4$$
$$+ 78X^8Y^6 + 78X^6Y^8 + 52X^4Y^{10} + 13X^2Y^{12} + 64XY + Y^{14}.$$

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- Modular equations of every degree exist, they are just hard to find
- We would like to find modular equations of even degree n

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- compute $A = \mathfrak{f}_1(\tau)$ and $B = \mathfrak{f}_1(2\tau)$ for a random τ in the upper half plane
- Find $\mathbb{Z}$-linear combination of terms $A^i B^j$ equal to zero

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- We LLL reduce the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \mathcal{R}(x_{0,0}) & \mathcal{I}(x_{0,0}) \\ 0 & 1 & 0 & \dots & 0 & \mathcal{R}(x_{0,1}) & \mathcal{I}(x_{0,1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & 1 & \mathcal{R}(x_{m,n}) & \mathcal{I}(x_{m,n}) \end{pmatrix}$$

LLL reduction

```
function lindep(V::Array{acb}, bits::Int)
  n = length(V)
  W = [ldexp(s, bits) for s in V]
  M = zero_matrix(ZZ, n, n + 2)
  for i = 1:n
    M[i, i] = ZZ(1)
    M[i, n + 1] = unique_integer(floor(real(W[i]) + 0.5))
    M[i, n + 2] = unique_integer(floor(imag(W[i]) + 0.5))
  end
  L = lll(M)
  return [L[1, i] for i = 1:n]
end
```

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- $$\Phi_2(X, Y) = X^2Y + 16X - Y^2$$

Computing modular equations

```
CC = ComplexField(128)
tau = CC(rand(), abs(rand()))
A = modweber_f1(tau)^8; B = modweber_f1(2*tau)^8

vals = [A^i*B^j for i in 0:2 for j in 0:2];
C = lindep(vals, 100)

R, (x, y) = PolynomialRing(ZZ, ["x", "y"])
Phi = sum([C[3*i+j+1]*x^i*y^j for i in 0:2 for j in 0:2])
```

Modular equation of degree 4

- Same polynomial $\Phi_2(X, Y)$ relates $\mathfrak{f}_1(\tau)^8$ and $\mathfrak{f}_1(2\tau)^8$ as relates $\mathfrak{f}_1(2\tau)^8$ and $\mathfrak{f}_1(4\tau)^8$.

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- Is there a relation between lower powers of $\mathfrak{f}_1(\tau)$ and $\mathfrak{f}_1(4\tau)$?
- Can proceed in the same way for degree $2n$, but the equations are enormous!

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- Can use the identities

$$\mathfrak{f}(\tau)^2\mathfrak{f}_1(\tau)^2\mathfrak{f}_2(\tau)^2 = 2$$

and

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  to yield:

- $$\frac{\mathfrak{f}_1(2\tau)^2\mathfrak{f}_1(6\tau)^2}{\mathfrak{f}_1(\tau)^2\mathfrak{f}_1(3\tau)^2} = \frac{4}{\mathfrak{f}_1(2\tau)^2\mathfrak{f}_1(6\tau)^2} + \mathfrak{f}_1(\tau)^2\mathfrak{f}_1(3\tau)^2.$$

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$$\mathfrak{f}_1(\tau)^8\mathfrak{f}_1(2\tau)^2\mathfrak{f}_1(4\tau)^4 + 8\mathfrak{f}_1(\tau)^4 - \mathfrak{f}_1(2\tau)^6\mathfrak{f}_1(4\tau)^4 = 0.$$

Modular equations of degree $2n$

- Can use a similar method to generate modular equations for degree $2n$ for all odd n
- Can even generate a modular equation of degree 4:

$$\mathfrak{f}_1(\tau)^8\mathfrak{f}_1(2\tau)^2\mathfrak{f}_1(4\tau)^4 + 8\mathfrak{f}_1(\tau)^4 - \mathfrak{f}_1(2\tau)^6\mathfrak{f}_1(4\tau)^4 = 0.$$

- Which can be rewritten:

$$\frac{\mathfrak{f}_1(\tau)^4}{\mathfrak{f}_1(2\tau)^2} + \frac{8}{\mathfrak{f}_1(2\tau)^4\mathfrak{f}_1(4\tau)^4} = \frac{\mathfrak{f}_1(2\tau)^2}{\mathfrak{f}_1(\tau)^4}.$$

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- Now there's enough data to guess at a general pattern.

General pattern for degree $4n$

- Define
$$A = \frac{\mathfrak{f}_1(\tau)^2}{\mathfrak{f}_1(4n\tau)^2}, \quad B = \frac{\mathfrak{f}_1(2\tau)^2}{\mathfrak{f}_1(2n\tau)^2}.$$

General pattern for degree $4n$

- Define
$$A = \frac{\mathfrak{f}_1(\tau)^2}{\mathfrak{f}_1(4n\tau)^2}, \quad B = \frac{\mathfrak{f}_1(2\tau)^2}{\mathfrak{f}_1(2n\tau)^2}.$$
- Search for linear combinations of $A^k B^l$ where
$$(8n-2)k + (4n-4)l \equiv m \pmod{24},$$
for fixed m.

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- The smallest equation we find is

$$B^{18} + 2B^{15} + 255B^{12} - 580B^9 + 255B^6 - 30B^3 + 1 = 256A^8B^{11} - 256A^4B^7 - 16\frac{B^{11}}{A^4} + \frac{B^{19}}{A^8}$$

```
CC = ComplexField(1000)
tau = CC(rand(), abs(rand()))
A = modweber_f1(tau)^2/modweber_f1(20*tau)^2
B = modweber_f1(2*tau)^2/modweber_f1(10*tau)^2

pairs = [p for p in
   Iterators.filter(x->mod(38*x[1]+16*x[2], 24) == 0,
      (k, l) for k in 0:24 for l in 0:24)]

vals = [A^k*B^l for (k, l) in pairs];
C = lindep(vals, 400)

Phi = sum(C[i]*x^pairs[i][1]*y^pairs[i][2]
    for i in 1:length(pairs))
```

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- Could just scale all exponents by factor of 24
- Chose to implement simple generic Puiseux series

The Puiseux series types

```
mutable struct PuiseuxSeriesRing{T <: RingElement} <: Ring
   laurent_ring::Ring
end

mutable struct PuiseuxSeries{T <: RingElement} <: RingElem
   data::LaurentSeries{T}
   scale::Rational{Int}
   parent::PuiseuxSeriesRing{T}

   function PuiseuxSeries{T}(d::LaurentSeries{T},
            scale::Rational{Int}) where T <: RingElement
      new{T}(d, scale)
   end
end
```

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- arithmetic operations, comparison operators
- powering, (exact) quotient
- in-place operators

Verifying the q-series identities

```
R, q = PuiseuxSeriesRing(ZZ, 1000, "q")

eta_qexp(q) = prod(1 - q^n for n = 1:50)*q^(1//24)
f1(q) = divexact(eta_qexp(q^(1//2)), eta_qexp(q))

A = divexact(f1(q)^2, f1(q^12)^2)
B = divexact(f1(q^2)^2, f1(q^6)^2)

A^8*(B^18 + 2B^15 + 255B^12 - 580B^9 + 255B^6
- 30B^3 + 1 - 256A^8*B^11 + 256A^4*B^7)
+ 16A^4*B^11 - B^19 == 0
```

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- Puiseux series store only denominator of exponents
- Use Flint's fast polynomial arithmetic for Laurent series over $\mathbb{Z}$
- Faster eta function q-series

Thank You

http://nemocas.org/